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1990 J. Phys. A: Math. Gen. 23 3647

(http://iopscience.iop.org/0305-4470/23/16/014)

J. Phys. A: Math. Gen. 23 (1990) 3647-3659. Printed in the UK

On unitary Lie superalgebras from the spin–orbit supersymmetrisation procedure

J Beckers†, N Debergh†, V Hussin‡ and A Sciarrino‡§

† Physique Théorique et Mathématique, Institut de Physique, B5, Université de Liège au Sart Tilman, B-4000, Liege 1, Belgium
‡ Centre de Recherches Mathématiques, Université de Montréal, CP 6128, Succ 'A', Montreal, Quebec H3C 3J7, Canada

Received 3 October 1989

Abstract. From the basis elements of a Clifford algebra Cl_n we generate a grading leading to a unitary Lie superalgebra when n is even. Such a construction is motivated by the understanding of the specific properties of the fermionic variables in the so-called spin-orbit coupling procedure of supersymmetrisation in $N=2$ supersymmetric quantum mechanics. The n-odd case is also considered and some specific examples are discussed.

1. Introduction

Bosonic *and* fermionic degrees of freedom are always simultaneously handled and mixed in operators defined in supersymmetric quantum mechanics. Their typical properties are specifically dependent on the supersymmetrisation procedure developed for the construction of the corresponding Hamiltonian.

Let us recall here that, in $N=2$ supersymmetric quantum mechanics, two procedures have often been used, that is the so-called standard (Witten 1981, 1982, de Crombrugghe and Rittenberg 1983) and the spin-orbit coupling (Balantekin 1985, Gamboa and Zanelli 1985, Kostelecky *et al* 1985, Beckers *et al* 1987, 1988) ones. They are subtended by the same usual properties on the *bosonic* variables but by different properties on the *fermionic* variables. When an n-dimensional harmonic oscillator is concerned, the two procedures have already been characterised (Beckers *et al* 1987, 1988) by general arguments (see section 2 for detailed information). Indeed the *standard* procedure implies that the $2n$ fermionic variables constitute (see equations (2.12)) the basis elements of a Clifford algebra Cl_{2n} while the *spin-orbit coupling* procedure handles $2n$ fermionic variables that cannot generate this Clifford algebra (see equations (2.12*b*) and (2.13)). In fact, the main purpose of this paper consists in the determination of the structure subtended by these last $2n$ fermionic variables of the spin-orbit coupling procedure. This requires the construction of the set

$$\{\alpha_k, \beta_k, \Xi^{kl} = -\Xi^{lk}, k, l = 1, 2, \ldots, n\}$$

starting from the basis elements of a Clifford algebra Cl_n. In this way we generate a grading leading to a unitary Lie superalgebra (Kac 1977a, b, 1978, Rittenberg 1978,

§ Present and permanent address: Dipartimento di Scienze Fisiche, Mostra d'Oltremare, Pad 19, I-80125, Napoli, Italy.

0305-4470/90/163647+13$03.50

Bars 1984, Hurni and Morel 1986), an original construction relating Clifford algebras and Lie superalgebras.

The content of this paper is distributed as follows. In section 2 we review the two supersymmetrisation procedures that are of special interest and we give some specific relations in each context with respect to the fermionic variables. Section 3 deals more particularly with the spin-orbit coupling procedure when the number n of spatial dimensions is *even*. In fact, this leads to a general construction (section 3.1) of a unitary Lie superalgebra by starting from a Clifford algebra Cl_n and permits a very simple matrix realisation (section 3.2). In section 4 we discuss the corresponding context but when n is *odd* and we develop, in section 5, some examples corresponding to the specific values $n=4$, (3), 6 and 8. Finally we present a few remarks and comments in section 6.

2. Supersymmetrisation procedures in quantum mechanics

As pointed out in the introduction different supersymmetrisation procedures have already been proposed and used in supersymmetric quantum mechanics. The so-called *standard* and *spin-orbit coupling* procedures can be uniquely characterised by structure relations on the fermionic quantities. Let us just give in this section such typical information (Beckers *et al* 1987, 1988).

By definition (Witten 1981, 1982), a quantum mechanical system is said to be N-supersymmetric if there exist N operators Q^a ($a=1,2,\ldots,N$), called supercharges, commuting with the Hamiltonian H

$$[Q^a, H]=0 \tag{2.1}$$

and satisfying the anticommutation relations

$$\{Q^a, Q^b\}=\delta^{ab}H. \tag{2.2}$$

The $N=1$ and $N=2$ contexts cover an important part of the interesting physical applications (D'Hoker *et al* 1988) and hereafter we limit ourselves to the $N=2$ supersymmetry. This is very often expressed in terms of conjugate supercharges defined by

$$Q_\pm=\frac{1}{\sqrt{2}}(Q^1\pm \mathrm{i}Q^2) \tag{2.3}$$

and then the corresponding $N=2$ superalgebra (2.1) and (2.2) reads

$$\{Q_+, Q_-\}=H \qquad \{Q_\pm, Q_\pm\}=0 \qquad [Q_\pm, H]=0. \tag{2.4}$$

In an n-dimensional space, the supercharges can be explicitly given by

$$Q_\pm=\frac{1}{\sqrt{2}}(p_j\mp \mathrm{i}\partial_j W(x))\xi_{\pm,j} \tag{2.5}$$

where the summation over the repeated index $j=1,2,\ldots,n$ is understood and where the operators p_j and x_j refer to the $(2n)$ bosonic degrees of freedom and satisfy, as usual, the relations

$$[p_i, x_j]=-\mathrm{i}\delta_{ij} \qquad p_j=-\mathrm{i}\partial_j\equiv-\mathrm{i}\frac{\partial}{\partial x_j} \qquad \boldsymbol{x}\equiv(x_1, x_2,\ldots,x_j,\ldots,x_n). \tag{2.6}$$

The so-called superpotential $W(\boldsymbol{x})$ is any scalar function of $\boldsymbol{x}$. The operators $\boldsymbol{\xi}_\pm$ refer to the fermionic degrees of freedom, which are supposed to satisfy in general the following relations (Beckers *et al* 1987, 1988):

$$\{\xi_{\pm,k}, \xi_{\pm,l}\} = 0 \qquad \forall k, l = 1, \ldots, n \tag{2.7}$$

$$\{\xi_{+,k}, \xi_{-,l}\} = \delta_{kl}\mathbb{1} - 2\mathrm{i}\Xi^{kl} \tag{2.8}$$

$$\Xi^{kl} = -\Xi^{lk} \qquad \Xi^\dagger = \Xi. \tag{2.9}$$

The corresponding supersymmetric Hamiltonian defined by the anticommutator of Q_+ and Q_- as in equation (2.4) is given by

$$H = \tfrac{1}{2}[\boldsymbol{p}^2 + (\boldsymbol{\nabla} W)^2] + \tfrac{1}{2}(\partial_k\partial_l W)[\xi_{+,k}, \xi_{-,l}] - [(\partial_k W)p_l - (\partial_l W)p_k]\Xi^{kl} \tag{2.10}$$

where we recognise the usual bosonic part with a potential term given by $U(\boldsymbol{x}) \equiv \frac{1}{2}(\boldsymbol{\nabla} W)^2$, the other part being in general dependent on both $\boldsymbol{x}$ and $\boldsymbol{p}$ as well as on the fermionic variables.

The above relations can clearly distinguish the two types of supersymmetrisation procedures (Beckers *et al* 1987, 1988).

If the relations (2.8) are such that all the Ξ^{kl} are equal to zero, we are dealing with the '*standard supersymmetrisation procedure*' where the number $(2n)$ of bosonic and fermionic degrees of freedom is the same.

Let us point out that, if we define the new $(2n)$ fermionic quantities (Beckers *et al* 1987)

$$\alpha_k = \xi_{+,k} + \xi_{-,k} \qquad \beta_k = \mathrm{i}(\xi_{-,k} - \xi_{+,k}) \tag{2.11}$$

we get from equations (2.7)-(2.9)

$$\{\alpha_k, \alpha_l\} = 2\delta_{kl}\mathbb{1} \qquad \{\beta_k, \beta_l\} = 2\delta_{kl}\mathbb{1} \tag{2.12a}$$

and

$$\{\alpha_k, \beta_l\} = 0 \tag{2.12b}$$

showing that the α_k and β_k generate a Clifford algebra Cl_{2n}.

If the Ξ^{kl} are not identically equal to zero in the equations (2.8), this case refers to the '*spin-orbit coupling supersymmetrisation procedure*', which is evidently only meaningful for $n \geqslant 2$. Originally considered in an $(n=3)$-dimensional space (Balantekin 1985), it is easy to see that the last term in the Hamiltonian (2.10) is essentially a spin-orbit coupling term when the (super) potential is a central one in particular.

Here we can also use the new quantities (2.11) and get the same equations (2.12*a*) *but*

$$\{\alpha_k, \beta_l\} = 4\Xi^{kl} = -\{\beta_k, \alpha_l\}. \tag{2.13}$$

As already mentioned in the introduction, we want to construct the set $\{\alpha_k, \beta_k, \Xi^{kl}, k, l = 1, \ldots, n\}$ according to equations (2.12*a*) and (2.13) for *arbitrary* $n \geqslant 2$.

Within such a program developed in the following sections, let us already mention a trivial case that we shall exclude, that is when the α_k and β_k are supposed to be linearly dependent. In fact, in such a case, it is easy to see that all the Ξ^{kl} are multiples of the identity and that the α_k and β_k satisfy two unitarily equivalent Clifford algebras

$$\beta_k = U\alpha_k U^\dagger \qquad k = 1, \ldots, n \qquad U^\dagger = U^{-1}. \tag{2.14}$$

This exclusion will necessarily fix n to be greater than two in our developments. Indeed, for $n=2$, the Clifford algebra Cl_2 of the elements α_1 and α_2 can be realised in terms of the 2×2 Pauli matrices

$$\alpha_1=\sigma_1 \qquad \alpha_2=\sigma_2. \tag{2.15}$$

Then, the relations (2.13) will be satisfied *if and only if* the unitary 2×2 matrix U takes the form

$$U=\begin{pmatrix}\exp(\mathrm{i}\varphi) & 0\\ 0 & \exp(-\mathrm{i}\varphi)\end{pmatrix} \qquad \varphi=(2p+1)\frac{\pi}{4} \tag{2.16}$$

where p is an integer. It leads through (2.14) and (2.15) to the other equivalent Clifford algebra(s)

$$\beta_1=\pm\sigma_2 \qquad \beta_2=\mp\sigma_1 \tag{2.17}$$

so that

$$\Xi^{12}=-\Xi^{21}=\mp\tfrac{1}{2}\mathbb{1}. \tag{2.18}$$

As a last remark, let us point out that, in such an $n=2$ case, the only independent elements, σ_1 and σ_2, generate together with the identity the Lie superalgebra su(1/1) (Kac 1977a, b, 1978).

3. Unitary superalgebras from Clifford algebras in *n* even dimensions

In this section, let us study the implications of the spin-orbit supersymmetrisation procedure in *n even* dimensions. More precisely, let us construct for each fixed n a unitary superalgebra from the basis elements of the Clifford algebra Cl_n according to our equations (2.12*a*) and (2.13). In the subsection 3.1 we adopt a general point of view while in the subsection 3.2 we use an explicit matrix realisation of the Clifford algebra, so leading to the usual matrix form of the superalgebra.

3.1. General construction

Let us start with $n=2m$ basis elements α_k generating a Clifford algebra Cl_n of dimension 2^n

$$\{\alpha_k, \alpha_l\}=2\delta_{kl}\mathbb{1} \qquad k, l=1, \ldots, n. \tag{3.1}$$

As we shall show, these basis elements transform as the n-dimensional vectorial representation of an algebra so(n). In the n-dimensional Euclidean space on which this so(n) operates, we define the *dual* elements (Bacry 1967) denoted by α^j ($j=1, \ldots, n$) obtained by using the completely antisymmetric Levi-Civita tensor ($\varepsilon^{12\ldots n}=1$)

$$\alpha^j=\frac{\mathrm{i}^r}{(n-1)!}\,\varepsilon^{jkl\ldots p}\alpha_k\alpha_l\ldots\alpha_p=\mathrm{i}^r(-1)^{j+1}\alpha_1\ldots[\alpha_j]\ldots\alpha_n \tag{3.2}$$

where $k<l<\ldots<p$ ($k, l, \ldots, p\neq j$), $r=1$ ($r=2$) if m is even (odd) and where the notation $[x]$ inside a product means that the factor x is missing. This set of dual elements is unitarily equivalent to the preceding one. Indeed, we have

$$\alpha^j=U\alpha_jU^\dagger \tag{3.3}$$

with

$$U = \tfrac{1}{2}\{(1+\mathrm{i})\mathbb{1} + [1+(-1)^{m+1}\mathrm{i}]\Lambda\} \tag{3.4}$$

where Λ is the so-called canonical element

$$\Lambda = \prod_{j=1}^{n} \alpha_j. \tag{3.5}$$

The proof is direct if we point out the properties

$$\Lambda^{\dagger} = (-1)^{r+1}\Lambda \qquad \Lambda^2 = (-1)^{r+1}\mathbb{1} \tag{3.6}$$

and

$$\alpha_j \Lambda = -\Lambda \alpha_j = (-1)^{j+1}\alpha_1 \ldots [\alpha_j] \ldots \alpha_n = (-\mathrm{i})^r \alpha^j. \tag{3.7}$$

The dual basis elements α^j satisfy

$$\{\alpha^j, \alpha^k\} = 2\delta^{jk}\mathbb{1} \tag{3.8a}$$

and

$$\{\alpha_j, \alpha^k\} = 4\Xi^{jk} = -\{\alpha_k, \alpha^j\}. \tag{3.8b}$$

The relations (3.8a) are trivially established from equations (3.1) and (3.3) while the antisymmetry of the equations (3.8b) is directly proved by using equations (3.7). Therefore, the basis elements α^j can be identified with our elements β_j introduced in section 2 in connection with the spin-orbit context. The $\frac{1}{2}n(n-1)$ quantities Ξ^{jk} will be formed by products of $(n-2)\alpha_j$ and their explicit form is given by ($l, p, \ldots, q \neq j, k$; $l < p < \ldots < q, j < k$)

$$\begin{aligned}\Xi^{jk} &= \tfrac{1}{2}\mathrm{i}^r(-1)^{j+k}\alpha_l\alpha_p \ldots \alpha_q \\ &= \tfrac{1}{2}\mathrm{i}^r(-1)^{j+k}\alpha_1 \ldots [\alpha_j] \ldots [\alpha_k] \ldots \alpha_n.\end{aligned} \tag{3.9}$$

Their commutation relations are easily determined (when $i \leqslant j \leqslant k \leqslant l$):

$$[\Xi^{ij}, \alpha_k] = 2\alpha^{ijk} \tag{3.10a}$$

$$[\Xi^{ij}, \alpha^k] = 2\alpha_{ijk} \tag{3.10b}$$

$$[\Xi^{ij}, \Xi^{kl}] = -(\delta_{il}\Xi_{jk} + \delta_{jk}\Xi_{il} + \delta_{ik}\Xi_{lj} + \delta_{jl}\Xi_{ki}) \tag{3.10c}$$

where

$$\alpha^{ijk} = \tfrac{1}{2}\mathrm{i}^r(-1)^{i+j+k}\alpha_1 \ldots [\alpha_i] \ldots [\alpha_j] \ldots [\alpha_k] \ldots \alpha_n \tag{3.11}$$

$$\alpha_{ijk} = \tfrac{1}{2}\alpha_i\alpha_j\alpha_k \tag{3.12}$$

and

$$\Xi_{ij} = \tfrac{1}{2}\alpha_i\alpha_j. \tag{3.13}$$

Now it is useful to compute the commutation relations of the Ξ_{ij}:

$$[\Xi_{ij}, \Xi_{kl}] = \delta_{il}\Xi_{jk} + \delta_{jk}\Xi_{il} + \delta_{ik}\Xi_{lj} + \delta_{jl}\Xi_{ki} \tag{3.14}$$

$$[\Xi_{ij}, \alpha_k] = \delta_{jk}\alpha_i - \delta_{ik}\alpha_j \tag{3.15a}$$

$$[\Xi_{ij}, \alpha^k] = \delta_{jk}\alpha^i - \delta_{ik}\alpha^j \tag{3.15b}$$

$$[\Xi_{ij}, \Xi^{kl}] = \delta_{il}\Xi^{jk} + \delta_{jk}\Xi^{il} + \delta_{ik}\Xi^{lj} + \delta_{jl}\Xi^{ki}. \tag{3.16}$$

Equations (3.14) are the usual commutation relations of the generators of so(n) and equations (3.15a, b) show that the α_i (α^i) transform as the vectorial (dual vectorial) representation of so(n) as was stated before. Moreover, equation (3.16) shows that the Ξ^{kl} also transform as a tensorial representation of so(n) and equation (3.10c) shows that their commutators give back the so(n) generators.

For $n=4$, these results correspond to the compact version so(4) of the Lorentz algebra. This case shows that the Ξ^{kl} and Ξ_{kl} are linearly dependent. In fact we have

$$\Xi^{kl}=-\tfrac{1}{2}\mathrm{i}\varepsilon^{klpq}\Xi_{pq} \qquad k,l,p,q=1,2,3,4. \tag{3.17}$$

For $n>4$, they are a generalisation of the analogue of the Lorentz algebra where the Ξ_{kl} have taken the place of the rotation generators and the Ξ^{kl} the place of the Lorentz boosts. In complete analogy with the case of the Lorentz algebra we can build up two commuting so$(n)_\pm$ ($n>4$) algebras spanned by

$$J^{\pm}_{kl}=\tfrac{1}{2}\mathrm{i}(\Xi_{kl}\pm\mathrm{i}\Xi^{kl}). \tag{3.18}$$

Now we shall call odd (respectively, even) elements the ones formed by odd (respectively even) products of α. By noticing that

$$\{\alpha^i,\alpha^{ijk}\}=-2\Xi_{jk} \qquad \text{(no summation)} \tag{3.19}$$

we conclude that we build up the even elements Ξ_{jk} and Ξ^{jk} by anticommuting odd elements (see equations (3.8b) and (3.19)). Moreover, we can easily see that, by anticommuting the odd elements α_i with α^{jkl}, we get even elements formed by products of $(n-4)$ α, which can be called Ξ^{ijkl}. By commuting these even Ξ^{ijkl} with the α_p, we get odd elements formed by $(n-5)$ α and called α^{ijklp}. Then, by anticommuting the α^i with the α^{ijklp}, we get the even element Ξ_{jklp}. Going on with this procedure, we can obtain all the Clifford algebra elements except $\Lambda\equiv(3.5)$. In such a construction we have always made use of the anticommutators between odd elements and of the even–even and even–odd commutators. Thus, finally, we have introduced a grading. The general structure of the grading is the following one. Let us call P the set of p labels $(i_1,i_2,\ldots,i_p)$, G_P the set formed by the products $g_p=\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_p}$ of p α and G^P its dual. For $p=0$, $G_P=G^P=\mathbb{1}$. Let us introduce the generalised commutation relation

$$\{g_P,g_Q]=g_Pg_Q-(-1)^{p.q}.g_Qg_P \qquad g_P\in G_P. \tag{3.20}$$

Equation (3.20) is not vanishing only if the sets P and Q have an *odd* number of equal labels. It follows the grading

$$\{G_P,G_Q]\subset G_R \tag{3.21a}$$

$$\{G^P,G^Q]\subset G_R \tag{3.21b}$$

$$\{G_P,G^Q]\subset G^R \tag{3.21c}$$

where R is the set of $(p+q-2k)$ labels formed by the union of the different labels in P and Q, k being the number of labels contained in $P\cap Q$.

In order that the generalised commutation relation be not vanishing k must be *odd* in equations (3.21a) and (3.21b) while in equation (3.21c) the number of labels in P not contained in Q must be *odd*.

Let us show that this grading leads to the superalgebra su(M/M) where $M=2^{m-1}$.

We can see by iteration of equations (3.14)–(3.16) that the elements formed by products of k α ($3\leqslant k\leqslant m$) transform under the action of Ξ_{ij} as the k-fold antisymmetric representation and for $m<k<n-2$ as its dual $(n-k)$-fold antisymmetric representation.

Moreover, we can take suitable combinations of even elements formed by k ($\neq 2$) and $(n-k)$ α such that they transform as the k-fold antisymmetric representation of $\mathrm{so}(n)_+$ (respectively, $\mathrm{so}(n)_-$) and the trivial representation of $\mathrm{so}(n)_-$ (respectively, $\mathrm{so}(n)_+$). For example, let us mention that these linearly independent combinations of even elements with 4 and $(n-4)$ α are, respectively, $(J^+_{ij}J^+_{lm})$ and $(J^-_{ij}J^-_{lm})$.

In general these combinations are products of $\frac{1}{2}k$ generators J^+ (J^-); they transform between themselves under the action of J^+_{pq} (J^-_{pq}) and commute with J^-_{pq} (J^+_{pq}). Thus, the elements

$$\begin{aligned}&(J^+_{ij}), (J^+_{ij}J^+_{lm}), \ldots, (J^+_{ij}J^+_{lm}\ldots J^+_{pq})\\&(J^-_{ij}), (J^-_{ij}J^-_{lm}), \ldots, (J^-_{ij}J^-_{lm}\ldots J^-_{pq})\end{aligned} \tag{3.22}$$

form two commuting sets. Each term of these sets explicitly gives an even antisymmetric representation of one $\mathrm{so}(n)$. All the terms together span an algebra $\mathrm{su}(M)$ where $M=2^{m-1}$ corresponds to the dimension of the fundamental *spinorial* representation of the $\mathrm{so}(n)$. In this way, to our two commuting sets, we associate the direct sum $\mathrm{su}(M)\oplus\mathrm{su}(M)$. This direct sum corresponds, up to one $\mathrm{u}(1)$, to the even part of the superalgebra $\mathrm{su}(M/M)$. Moreover, by noticing that the anticommutators of the odd elements give the even part, it follows that suitable combinations of these odd elements must transform as the sum of the representations $(M, \bar{M})$ and $(\bar{M}, M)$ (Kac 1978). Thus, we conclude that we get the superalgebra $\mathrm{su}(M/M)$.

3.2. Matrix realisation

Let us recall a few essential properties of the superalgebra $\mathrm{su}(M/M)$. It can be written (Kac 1977a, b) in a matrix notation in the following form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.23}$$

where A, B, C and D are $M\times M$ complex matrices satisfying

$$A^\dagger = A \qquad D^\dagger = D \qquad \mathrm{Tr}\,A = \mathrm{Tr}\,D \qquad \text{and} \qquad B^\dagger = C. \tag{3.24}$$

The block diagonal matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \tag{3.25}$$

span the algebra $\mathrm{su}(M)\oplus\mathrm{su}(M)\oplus\mathrm{u}(1)$ and they form the even part $\mathrm{su}(M/M)_{\bar{0}}$ of the superalgebra. The off-diagonal block matrices, which form the odd part, $\mathrm{su}(M/M)_{\bar{1}}$

$$\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \tag{3.26}$$

transform as the sum of the representations $(M, \bar{M})$ and $(\bar{M}, M)$ of $\mathrm{su}(M)\oplus\mathrm{su}(M)$, and their anticommutators span the even part (the Lie algebra).

The dimension of the superalgebra is for the even part equal to $2M^2-1$ and for the odd part equal to $2M^2$. This superalgebra is not simple as it contains the one-dimensional ideal consisting of the $2M$-dimensional identity matrix.

We have shown in subsection 3.1 that the superalgebra $\mathrm{su}(M/M)$ (with $M=2^{m-1}$) can be related to a Clifford algebra Cl_n with $n=2m$ basis elements. In subsection 3.1 the construction was presented in some abstract way; here let us show explicitly how to realise the matrix form of su(M/M) in terms of the n basis elements of Cl_n and their products. Indeed, these elements can be written in the following matrix form (Kostelecky *et al* 1985, Beckers *et al* 1988)

$$\alpha_\mu = \begin{pmatrix} 0 & B_\mu \\ B_\mu & 0 \end{pmatrix} = B_\mu \otimes \sigma_1 \qquad \mu = 1, \ldots, n-1$$
$$\alpha_n = \begin{pmatrix} 0 & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & 0 \end{pmatrix} = \mathbb{1} \otimes \sigma_2 \tag{3.27}$$

where the B_μ are $2^{m-1} \times 2^{m-1}$ Hermitian matrices, basis elements of a Clifford algebra Cl_{n-1} and where $\mathbb{1}$ is the corresponding identity matrix. An explicit realisation of the B_μ is obtained by setting (Sattinger and Weaver 1986)

$$B_\mu = \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \underset{(\mu)}{\sigma_1} \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2$$
$$B_{m+\mu-1} = \sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2 \tag{3.28}$$
$$B_{n-1} = \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3$$

for $1 \leqslant \mu \leqslant m-1$. The elements α_k in equation (3.27) then have the form of equation (3.26).

Now we introduce n elements β_k satisfying equations (2.12a) and (2.13), which are defined by

$$\beta_k = U \alpha_k U^\dagger \tag{3.29}$$

where the U-operator is given by equation (3.4). Here $\Lambda \equiv (3.5)$ will be of the form

$$\Lambda = \mathrm{i}\left(\prod_{k=1}^{n-1} B_k\right) \otimes \sigma_3 \tag{3.30}$$

and, since the B_μ span Cl_{n-1}, it can be shown that we have

$$\Lambda = -\mathbb{1} \otimes \sigma_3 \qquad \text{for } m \text{ even}$$
$$\Lambda = \mathrm{i}\mathbb{1} \otimes \sigma_3 \qquad \text{for } m \text{ odd.} \tag{3.31}$$

It follows from equations (3.4) and (3.31) that

$$U = \begin{pmatrix} \mathrm{i}\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{3.32}$$

Then we have from equations (3.29) and (3.32)

$$\beta_\mu = -B_\mu \otimes \sigma_2 \qquad \beta_n = \mathbb{1} \otimes \sigma_1. \tag{3.33}$$

Evidently we see that equation (3.33) gives a matrix expression for the elements $\alpha^j \equiv (3.3)$.

Making the anticommutators of the α_i and β_j we obtain the expression for the Ξ^{ij} (cf equation (2.13)):

$$\Xi^{\mu\nu} = -\tfrac{1}{4}\mathrm{i}[B_\mu, B_\nu] \otimes \sigma_3 \qquad \Xi^{\mu n} = \tfrac{1}{2} B_\mu \otimes \mathbb{1}_2. \tag{3.34}$$

Equation (3.34) shows that the Ξ^{ij} have the form of equation (3.25). Equation (3.24) follows from the Hermiticity property of the B_μ matrices and from the fact that $\mathrm{tr}[B_\mu, B_\nu]=0$.

From the commutation of the Ξ^{ij} with the α_k or β_k, we generate new odd elements that can be expressed as a product of an odd number of α_i. From equation (3.27) one can show that all such elements have the form of equation (3.26). The even elements obtained by their anticommutator will take the form of equation (3.25) and satisfy equation (3.24). In particular, the $\Xi_{ij}\equiv(3.13)$ are realised in the form

$$\Xi_{\mu\nu}=\tfrac{1}{4}[B_\mu, B_\nu]\otimes\mathbb{1}_2 \qquad \Xi_{\mu n}=\tfrac{1}{2}\mathrm{i}B_\mu\otimes\sigma_3. \tag{3.35}$$

So we have obtained the matrix structure of $\mathrm{su}(M/M)$.

To get a better insight into this structure and to find explicitly some of the results of subsection 3.1 let us compute the elements $J^{\pm}_{ij}\equiv(3.18)$. We have

$$\begin{aligned} J^+_{\mu\nu}&=\frac{\mathrm{i}}{2}\begin{pmatrix} B_\mu B_\nu & 0\\ 0 & 0\end{pmatrix} \qquad & J^+_{\mu n}&=\frac{1}{2}\begin{pmatrix} -B_\mu & 0\\ 0 & 0\end{pmatrix}\\ J^-_{\mu\nu}&=\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 0\\ 0 & B_\mu B_\nu\end{pmatrix} \qquad & J^-_{\mu n}&=\frac{1}{2}\begin{pmatrix} 0 & 0\\ 0 & B_\mu\end{pmatrix}. \end{aligned} \tag{3.36}$$

From equations (3.36) one can see at once all the properties we have stated in subsection 3.1.

4. The odd case

Let us discuss here the case of $(n\pm1)$ odd dimensions with n even. In this case we cannot generate a structure of a unitary superalgebra starting from the generators of a Clifford algebra $\mathrm{Cl}_{n\pm1}$ and including our structure (2.12*a*) and (2.13). There are several arguments to realise that our construction cannot work.

If we start from the remark that the α_i ($i=1,\ldots,p$ odd) form a basis of a vectorial representation and if we construct the dual basis α^i, we have

$$\{\alpha^k, \alpha_k\}=2\mathrm{i}^r\alpha_1\ldots\alpha_p\neq0 \qquad \text{(no summation)} \tag{4.1}$$

so the condition (3.8*b*) cannot be satisfied.

If we analyse the structure of an odd-dimensional Clifford algebra, there are two ways of generating this algebra. The first way is, for example, to start from an even-dimensional Clifford algebra with basis α_i ($i=1,\ldots,n$) and to add a further element

$$\alpha_{n+1}=(-\mathrm{i})^{r+1}\prod_{k=1}^{n}\alpha_k=(-\mathrm{i})^{r+1}\Lambda. \tag{4.2}$$

Using the operator (3.4) we have $\beta_i=\alpha^i\equiv$ equation (3.3), $i=1,\ldots,n$ and

$$\beta_{n+1}=U\alpha_{n+1}U^\dagger=\alpha^{n+1}=\alpha_{n+1}=(-\mathrm{i})^{r+1}\Lambda \tag{4.3}$$

and therefore we have

$$\{\beta_{n+1}, \alpha_k\}=0 \qquad \{\beta_{n+1}, \alpha_{n+1}\}\neq0 \tag{4.4}$$

so that equation (2.13) cannot be satisfied.

The second way is to construct the Clifford algebra Cl_{n-1} as a substructure of Cl_n with a basis of $(n-1)$ α_i generators. The objects Ξ^{ij} are built up by the anticommutators of the α_i with the $\beta_j = U\alpha_j U^\dagger$ $(i, j = 1, \ldots, n-1)$ where $U \equiv$ equation (3.4). The commutators of the Ξ^{ij} generate new elements that cannot be obtained as products of the α_i $(i = 1, \ldots, n-1)$ and then cannot belong to our Clifford algebra Cl_{n-1}. In fact, they belong to Cl_n as expected by the preceding results. Finally, by a simple computation we can see that the number of elements that can be obtained as products of α_j $(j = 1, \ldots, n-1)$ is different from the number of elements of a superalgebra.

Evidently this does not mean that we cannot supersymmetrise in the odd-dimensional case. Indeed, one can perform the spin-orbit coupling supersymmetrisation procedure just by starting with an $(n-1)$-dimensional subset of the n-even basis elements α_k of a Clifford algebra Cl_n. The corresponding $(n-1)$ α^j given by equation (3.3) are evidently our β_j, which clearly satisfy equation (2.13). The Ξ^{kj} $(k, j = 1, \ldots, n-1)$ form a subset in the Clifford algebra Cl_n.

5. The explicit examples: $n = 4$, 6 and 8

Let us present here some explicit examples in order to show how our construction works. Subsection 5.1 deals with the important $n = 4$ case, to which the physical case $n = 3$ is directly related. Subsections 5.2 and 5.3 develop the $n = 6$ and $n = 8$ cases, respectively. Both cases lead to a direct generalisation to arbitrary even n. They also show some differences with respect to the $n = 4$ case.

5.1. *The case $n = 4$*

Starting with the four basis elements $\alpha_1, \ldots, \alpha_4$ of a Clifford algebra Cl_4 we evidently generate by products sixteen independent elements. Fifteen of them can be used to generate the su(2/2) superalgebra. The six products of two α generate an so(4) algebra. Its isomorphism to su(2)$\oplus$su(2) is clearly seen using the linear combinations

$$J_a^{\pm} = -\tfrac{1}{4}\mathrm{i}(\tfrac{1}{2}\varepsilon_{abc}\alpha_b\alpha_c \pm \alpha_4\alpha_a) \qquad a, b, c = 1, 2, 3. \tag{5.1}$$

They generate together with the identity the even part $\mathrm{su}(2/2)_{\bar{0}} \equiv \mathrm{su}(2)\oplus\mathrm{su}(2)\oplus\mathrm{u}(1)$ of su(2/2). We can see that the four basis elements α_k correspond to the vectorial representation of so(4) and that the four products of three α correspond, up to the imaginary factor i, to the dual vectorial representation. They form the odd part $\mathrm{su}(2/2)_{\bar{1}}$. Let us insist on the fact that $\Lambda = \Pi_{i=1}^4 \alpha_i$ is not included in su(2/2) since it cannot be obtained by the grading we have defined.

The equations (2.12*a*) and (2.13) are evidently satisfied by the $\beta_k \equiv \alpha^k$ ($\equiv$equation (3.2)) elements $(k = 1, \ldots, 4)$ and by the Ξ^{kl} defined according to equation (3.9) and given explicitly by

$$\Xi^{ab} = \tfrac{1}{2}\mathrm{i}\varepsilon_{abc}\alpha_4\alpha_c \qquad \Xi^{a4} = -\tfrac{1}{4}\mathrm{i}\varepsilon_{abc}\alpha_b\alpha_c. \tag{5.2}$$

This clearly shows the relation (3.17) between the Ξ^{kl} and Ξ_{kl}.

Finally, it is easy to see that an explicit matrix realisation of the α is given by

$$\alpha_a = \sigma_a \otimes \sigma_1 \qquad \alpha_4 = \mathbb{1} \otimes \sigma_2 \tag{5.3}$$

in accordance with equations (3.25) and (3.26). The β, which are unitarily equivalent to the α, are given (cf equation (3.33)) by

$$\beta_a = -\sigma_a \otimes \sigma_2 \qquad \beta_4 = \mathbb{1} \otimes \sigma_1. \tag{5.4}$$

The Ξ satisfying equation (2.13) are then obtained as

$$\Xi^{ab}=\tfrac{1}{2}\varepsilon_{abc}(\sigma_c\otimes\sigma_3) \qquad \Xi^{a4}=\tfrac{1}{2}(\sigma_a\otimes\mathbb{1}_2). \tag{5.5}$$

Let us now come to the $n=3$ case in order to see how it is a subcase of the $n=4$ case. In fact, we have to find three α and three β satisfying the structure of equations (2.12*a*) and (2.13). As stated before (see section 4) we cannot do this starting with a Clifford algebra Cl_2 and realising Cl_3 by adding an element. Instead we have to go to the algebra Cl_4 and consider a subset of three α, for example, the α_a in equation (5.3), which form a Cl_3. The α_a together with the β_a defined in equation (5.4) and the Ξ^{ab} defined in equation (5.3) satisfy our structure as expected. This choice is directly related to the example of the three-dimensional harmonic oscillator with a spin-orbit coupling term (Balantekin 1985, Beckers *et al* 1987). We also have the following relations:

$$[\alpha_a,\Xi^{bc}]=\pm\mathrm{i}\alpha_4 \qquad a\neq b, c. \tag{5.6}$$

This means that we generate a new element α_4 that cannot be obtained as a product of the α_a. Thus, by this process, we are going out of our starting Cl_3 and generating our preceding su(2/2) superalgebra.

5.2. *The case $n=6$*

Now we take a 64-dimensional Clifford algebra Cl_6 with six α as basis elements. Let us identify the su(4/4) superalgebra we generate from Cl_6 and which is of dimension 63. First, the even part $\mathrm{su}(4/4)_{\bar{0}}\equiv\mathrm{su}(4)\oplus\mathrm{su}(4)\oplus\mathrm{u}(1)$ is formed by the identity and suitable combinations of the fifteen products of two α and the fifteen products of four α. The first fifteen elements generate a well known so(6) algebra. The second set transforms as the dual twofold antisymmetric representation under this so(6). Their suitable combinations (i.e. the $J^{\pm}\equiv(3.18)$) generate $\mathrm{so}(6)\oplus\mathrm{so}(6)$, which is isomorphic to $\mathrm{su}(4)\oplus\mathrm{su}(4)$. The odd part (32 dimensions) is formed by the α, their dual β and by the (20) products of three α. From the equations (3.8*b*) and (3.19) they generate the even part.

In this case (in comparison with the $n=4$ case) we see that the Ξ^{ij} ($\equiv$equation (3.9)) do not form an algebra. It is only the set of the Ξ^{ij} *and* Ξ_{ij} that form $\mathrm{so}(6)\oplus\mathrm{so}(6)$.

5.3. *The case $n=8$*

Now we deal with a Clifford algebra Cl_8 that is of dimension 256. The superalgebra su(8/8) is of dimension 255. Its even part $\mathrm{su}(8/8)_{\bar{0}}\equiv\mathrm{su}(8)\oplus\mathrm{su}(8)\oplus\mathrm{u}(1)$ (127 dimensions) is obtained as follows. The two independent combinations (see equation (3.18)) of an element belonging to the set (28 elements) formed by the products of two α (which generate a so(8)), with its dual belonging to the set formed by the products of six α, generate $\mathrm{so}(8)\oplus\mathrm{so}(8)$.

It is known that the adjoint representation of su(8) decomposes with respect to so(8) as $\underline{63}\rightarrow\underline{28}+\underline{35}$, where $\underline{28}$ is the adjoint representation of so(8) and $\underline{35}$ has to be identified with <u>one</u> of the three well known so(8) representations, that is the twofold symmetric one, the fourfold antisymmetric one or its conjugate according to the identification of the fundamental su(8) representation with correspondingly the vectorial one or the spinorial one or its conjugate of so(8). Thus, in order to recover su(8)⊕su(8), we have to add to so(8)⊕so(8) the seventy products of the four α in correspondence with the above antisymmetric representations. The odd part (128 dimensions) will be formed by the α, the products of three α and their duals.

6. Remarks and comments

The *physical* context associated with the spin-orbit coupling procedure of supersymmetrisation in quantum mechanics has thus led to an explicit construction of unitary Lie superalgebras from Clifford algebras. More precisely, we have shown how, from the Clifford algebra Cl_n, we can obtain the unitary superalgebra $\mathrm{su}(2^{m-1}/2^{m-1})$ when $n=2m$ and $n\geqslant 2$. In fact, the case $n=2$ can also be included as discussed in section 2 if we accept that the α and β are dependent. Let us insist on the fact that the knowledge of only the basis elements of Cl_n is sufficient in order to make the construction. These α_k $(k=1,\ldots,n)$ really are the cornerstone of our construction. This has also to deal with superderivations of a Clifford algebra (Scheunert 1979). Finally, let us also mention that the above construction shows in particular that only one Clifford element, the so-called canonical element, $\Lambda\equiv(3.5)$, has been omitted in the corresponding unitary superalgebra. The explanation is trivial if we notice that Λ is the only element that cannot be generated through the structure relations between odd and even operators. Nevertheless, let us mention that if we add Λ to the above generators we get the superalgebra $\mathrm{u}(2^{m-1}/2^{m-1})$.

As a final remark, let us come back on the symmetry properties of the supersymmetric Hamiltonian H given by equation (2.10). If the superpotential corresponds to central forces, rotational invariance will be included in the (super)symmetries of the system. Indeed, when $L_{kl}=x_kp_l-x_lp_k$, we have

$$[H, J_{kl}]=0 \qquad k,l=1,2,\ldots,n$$

where the operators

$$J_{kl}=L_{kl}-\mathrm{i}\Xi_{kl}$$

are total angular momentum operators generating a so(n) algebra. For $n=4$ *(and 3)*, such an invariance holds since the Ξ^{kl} inside the Hamiltonian are really associated with the elements of a so(4) algebra (see subsection 5.1) and consequently terms of the type $L_{kl}\Xi^{kl}$ are really spin-orbit coupling terms. For $n>4$, the Ξ^{kl} do not close under commutation but satisfy the relations (3.16), implying once again the conservation of the J_{kl}.

Acknowledgments

Two of us (J Beckers and A Sciarrino) would like to thank the CRM for its hospitality. The research of V Hussin is partially supported by research grants from NSERC of Canada and the 'Fonds FCAR du Gouvernement du Québec'.

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